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A PRECONDITIONED FORMULATION OF THE CAUCHY-RIEMANN EQUATIONS

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# A PRECONDITIONED FORMULATION OF THE CAUCHY-RIEMANN EQUATIONS

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## Abstract

A preconditioning of the Cauchy-Riemann equations which results in a second-order system is described. This system is shown to have a unique solution if the boundary conditions are chosen carefully. This choice of boundary condition enables the solution of the first-order system to be retrieved. A numerical solution of the preconditioned equations is obtained by the multigrid method.

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## Introduction

In this paper we are concerned with the Cauchy-Riemann equations. The two-dimensional form of these equations may be written as follows:

$$U_x + AU_y = B \quad \text{in } \Omega, \quad (1a)$$

$$U \cdot n = g \quad \text{on } \Gamma, \quad (1b)$$

where  $\Omega$  is some bounded domain with boundary  $\Gamma$  and

$$U^T = (u(x,y), v(x,y)),$$

$$B^T = (\rho(x,y), \xi(x,y)),$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The function  $\rho$  and the boundary data satisfy the compatibility condition

$$\iint_{\Omega} \rho \, d\Omega = \int_{\Gamma} g \, ds \quad (2)$$

which follows from Gauss' divergence theorem. This condition guarantees that the problem has a unique solution. We shall restrict ourselves to the case where  $\Omega$  is the unit square.

The Cauchy-Riemann equations occur in many problems in fluid mechanics. The velocity components  $u$  and  $v$  are usually known as the primitive variables. It has been found that the numerical solution of the Cauchy-Riemann equations can be enhanced by introducing a stream function, which

satisfies a Poisson equation. There are many fast, direct methods for solving such equations, see for example, Buzbee, Golub, and Nielson [2] and Hockney [6]. The emergence of multigrid methods means that there now exist fast and efficient iterative methods which are competitive with these direct methods, see for example, Foerster and Witsch [4]. However, the primitive variable formulation does possess advantages since it allows treatment of more general flows. Ghil and Balgovind [5] have proposed a direct method to solve for the primitive variables in a problem where these variables are periodic in the x-direction.

This work was motivated by an examination of a least squares formulation of the Cauchy-Riemann equations (Fix and Rose [8]). In two dimensions Gatski, Grosch and Rose [9] employ an iteration scheme due to Kaczmarz to solve the resulting problem but the method was found to be too inefficient to serve as a basis for a three-dimensional treatment of the incompressible Navier-Stokes equations. The method described here could provide the basis for a fast Cauchy-Riemann solver in three dimensions.

This technique provides insight into the treatment of more difficult problems. Johnson [10] has successfully used a closely related method to solve the full Euler equations in straight channel with a 10% half-thick circular arc airfoil mounted on its lower wall.

Here we describe a formulation of the problem which results in two Poisson equations, one for each of the primitive variables. Section 2 describes the preconditioning and shows that with suitable extra boundary conditions the new problem possesses a unique solution which satisfies the Cauchy-Riemann equations. Section 3 contains details of the solution procedure and describes the components of the multigrid method used. Section 4 contains numerical results and computational details.

## 2. Preconditioned Equations

We may write the system of equations (1) in the form

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \rho \\ \xi \end{pmatrix}, \quad (3)$$

i.e.,

$$D U = B. \quad (4)$$

We consider a preconditioning of this system obtained by premultiplying Eq. (4) by the matrix  $D^T$ . This yields the following equation:

$$(D^T D)U = D^T B \quad (5)$$

where

$$D^T D = \begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{pmatrix}$$

and

$$D^T B = \begin{pmatrix} \frac{\partial \rho}{\partial x} - \frac{\partial \xi}{\partial y} \\ \frac{\partial \rho}{\partial y} + \frac{\partial \xi}{\partial x} \end{pmatrix}.$$

The symbol  $\nabla^2$  represents the Laplacian operator  $\partial^2/\partial x^2 + \partial^2/\partial y^2$ . This preconditioning has led to a system of second-order equations and so for the problem to be well-posed we require an additional boundary condition. In the remainder of this section we show that it is natural to take  $v_x - u_y = \xi$  as the extra boundary condition. For simplicity we assume homogeneous boundary

data, i.e.  $U \cdot n = 0$  where  $n$  is the unit outward drawn normal to the domain. There is no loss of generality in doing so since  $U \cdot n = 0$  can always be achieved by an appropriate translation. In the following analysis we make the assumption that  $u, v, \rho$  and  $\xi$  have sufficiently many continuous derivatives.

### Theorem 1

The solution of the system of equations (5) with the conditions that  
 $U \cdot n = 0$  and  $v_x - u_y = \xi$  on the boundary of  $\Omega$  is unique.

Proof: Let  $u^{(1)}, v^{(1)}$  and  $u^{(2)}, v^{(2)}$  be two solutions of (5) with  $u^{(1)} \neq u^{(2)}$  and  $v^{(1)} \neq v^{(2)}$ . Let  $\phi = u^{(1)} - u^{(2)}$  and  $\psi = v^{(1)} - v^{(2)}$  then  $\phi$  and  $\psi$  satisfy the following

$$\nabla^2 \phi = 0, \quad \nabla^2 \psi = 0, \quad \text{in } \Omega,$$

$$\psi_x - \phi_y = 0, \quad \text{on } \Gamma,$$

$$\phi = 0, \quad \text{on } x = 0 \text{ and } x = 1,$$

$$\psi = 0, \quad \text{on } y = 0 \text{ and } y = 1.$$

The condition that  $\psi = 0$  along  $y = 0$  and  $y = 1$  implies that  $\phi_y = 0$  along these lines since  $\psi_x - \phi_y = 0$  there. Thus we obtain the following problem for  $\phi$ :

$$\phi_y = 0$$

$$\phi = 0$$

$$\nabla^2 \phi = 0$$

$$\phi = 0$$

$$\phi_y = 0$$

Figure 1

It follows by a maximum principle (Protter and Weinberger [7]) that  $\phi \equiv 0$  in  $\Omega$ . In a similar way it can be shown that  $\psi \equiv 0$  in  $\Omega$ . Thus the solution of the system (5) is unique.

## Theorem 2

The solution of the second-order system (5) satisfies Eq. (1).

Proof: The following identities hold:

$$\nabla^2 u - \frac{\partial \rho}{\partial x} + \frac{\partial \xi}{\partial y} \equiv \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y},$$

$$\nabla^2 v - \frac{\partial \rho}{\partial y} - \frac{\partial \xi}{\partial x} \equiv \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x},$$

where  $P = u_x + v_y - \rho$  and  $Q = v_x - u_y - \xi$ . We aim to show that  $P \equiv 0$  and  $Q \equiv 0$  in  $\Omega$ .

If  $U$  is a solution of the system (5) then

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0,$$

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0.$$

It follows, by differentiation, that  $\nabla^2 Q = 0$  in  $\Omega$ . On  $\Gamma$  we have the boundary condition  $Q = 0$ . Therefore  $Q \equiv 0$  in  $\Omega$  by the uniqueness of the solution of Laplace's equation with Dirichlet boundary conditions which follows from a maximum principle (see Protter and Weinberger [7]).

Now  $\partial P/\partial x = 0$  and  $\partial P/\partial y = 0$  in  $\Omega$  implies that  $P$  is a constant, with value  $M$  say, in  $\Omega$ .

Using Green's theorem we obtain

$$\iint_{\Omega} (u_x + v_y - \rho) dx dy = \int_{\Gamma} U \cdot n ds + \iint_{\Omega} \rho dx dy = 0. \quad (6)$$

The first term on the right-hand side of Eq. (6) is zero since  $U \cdot n = 0$  around  $\Gamma$  while the second term is zero by the compatibility condition. Therefore, since  $u_x + v_y - \rho = M$ , a constant, we must have  $M \equiv 0$ . Therefore,  $P \equiv 0$  in  $\Omega$ . We have thus shown that the solution of the system (5) satisfies the Cauchy-Riemann equations.

It is certainly true that the solution of the Cauchy-Riemann equations satisfies the system (5) since the latter can be viewed as being obtained by differentiation of the former. The results of this section carry over to three dimensions and these will be reported elsewhere.



### 3. Method of Solution

The system of equations (5) represents two Poisson equations for  $u$  and  $v$ . The problem has now been cast into a form, which is amenable to many of the standard iterative methods of solution. We have chosen to use the multigrid method since such methods are fast and efficient.

The domain  $\Omega$  is covered with a square grid of mesh length  $h = 1/N$  where  $N$  is a positive integer. Each of the Poisson equations:

$$\nabla^2 u = \frac{\partial \rho}{\partial x} - \frac{\partial \xi}{\partial y},$$

$$\nabla^2 v = \frac{\partial \rho}{\partial y} + \frac{\partial \xi}{\partial x},$$

are discretized using standard second-order central difference approximations. Along  $x = 0$  and  $x = 1$  we have Dirichlet boundary conditions for  $u$ . Along the boundaries  $y = 0$  and  $y = 1$  the finite difference formula is constructed by eliminating exterior imaginary points using a central difference approximation to the Neumann boundary condition  $u_y = v_x - \xi$ . Since  $v$  is given along  $y = 0$  and  $y = 1$  we know the value of  $v_x$  along these lines. Thus, we obtain a problem for  $u$  which does not contain any unknown values of  $v$ . Similarly, we obtain a problem for  $v$  which does not contain any unknown values of  $u$ . Therefore, the system for  $u$  and  $v$  becomes uncoupled and we can solve for the one independent of the other.

We consider a multigrid method of solution to this problem using the Correction Storage algorithm of Brandt [1]. Let  $G_1, \dots, G_M$  be a sequence of grids approximating the domain  $\Omega$  with corresponding mesh sizes  $h_1, \dots, h_M$ . Let  $h_k = 2h_{k+1}$  for  $k = 1, \dots, M-1$ . The problem is discretized on each grid  $G_k$  as described above. We describe below the components chosen in the multigrid procedure.

(1) Relaxation

Pointwise Gauss-Seidel relaxation is used with the points ordered in the checkerboard (even-odd) manner. This is one of the components of the "non-standard" multigrid techniques introduced by Foerster, Stüben, and Trottenberg [3]. At the end of the relaxation sweep the residuals on any grid  $G_k$  are zero at the odd points since at this stage all the odd-point equations are simultaneously satisfied.

(2) Fine-to-coarse Transfer

Checkerboard Gauss-Seidel produces highly oscillating residuals and so it is inadvisable to simply transfer the residuals by injection to a coarser grid. Instead we transfer the residuals by half-weighting to the coarse grid. If the points of the coarse grid are also even points on the finer grid then the transfer of the residuals reduces to half-injection.

(3) Coarse-to-fine Transfer

Bilinear interpolation is used to transfer the correction to the fine grid to provide a new approximation there.

We use a version of multigrid which begins with an approximation on the finest level of discretization and then cycles between grids until the convergence criterion has been attained. The decisions to switch grids are made automatically by internal checks in the algorithm. The decision to switch to a coarser grid is made if

$$\| \mathcal{R}^{k+1} \|_2 > \eta \| \mathcal{R}^k \|_2 ,$$

where  $\eta < 1$  is a positive constant and  $\mathcal{R}^k$  denotes the residuals of the discrete equations on grid  $G_k$ . The decision to return to a finer grid is

made according to the following criterion:

$$\|\mathcal{Q}^{k-1}\|_2 < \delta \|\mathcal{Q}^k\|_2 ,$$

where  $\delta < 1$  is a positive constant.

#### 4. Numerical Results

The multigrid method defined in the previous section was used to solve three test problems. These three problems are characterized as follows:

$$\begin{aligned} \text{(i)} \quad \rho &= 0, & \xi &= 0, \\ u &= x, & v &= -y; \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \rho &= 0, & \xi &= \left(\frac{\pi}{m}\right) (m^2 + n^2) \sin(n\pi x) \sin(m\pi y), \\ u &= \sin(n\pi x) \cos(m\pi y), & v &= -\left(\frac{n}{m}\right) \cos(n\pi x) \sin(m\pi y); \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \rho &= 0, & \xi &= [(\alpha^2 - n^2\pi^2)/\alpha] e^{-\alpha y} \sin(n\pi x), \\ u &= e^{-\alpha y} \sin(n\pi x), & v &= (n\pi/\alpha) e^{-\alpha y} \cos(n\pi x). \end{aligned}$$

The first problem is homogeneous with a linear solution. The driving function, boundary conditions and solution of the second problem are oscillatory and this behavior increases as  $m$  and  $n$  increase. The third problem presents a test of the multigrid method on a boundary layer problem.

In all of the numerical experiments we took the finest grid to be such that  $N = 32$  and considered a total of five grids in the multigrid context.

The initial approximation on  $G_M$  was taken to be zero everywhere except where the solution is specified by the boundary conditions. With this initial approximation, the initial residual  $\|P_0^M\|_2$  was computed and the convergence tolerance on  $G_M$  was chosen to be

$$\epsilon_M = 10^{-4} \|P_0^M\|_2.$$

We define a work unit to be the computational work in one relaxation sweep over the finest grid  $G_M$ . The value of the switching parameters  $\eta$  and  $\delta$  were chosen to be 0.6 and 0.2 respectively.

The results of the algorithm applied to  $u$  for problems (i), (ii), and (iii) are shown in Tables I, II, and III respectively. We give the number of work units required to attain the convergence criterion, the asymptotic convergence factor  $\bar{\lambda}$ , and the  $\ell_2$ -norm of the error in the numerical solution for  $u$  on  $G_M$ . Similar results were obtained when the algorithm was applied to the primitive variable  $v$  in the above cases and indeed the number of work units was the same in all cases. We have used the notation  $a \cdot b - c$  for  $a \cdot b \times 10^{-c}$ .

It can be seen that the method exhibits the usual multigrid behavior by examining the asymptotic convergence factors obtained. The accuracy of the discrete approximation to problem (ii) decreases as  $m$  and  $n$  increase as one might expect since the number of mesh points per wavelength of the solution decreases.

The approach described in this paper to precondition the Cauchy-Riemann equations together with the given choice of boundary conditions, which is based on firm theoretical results, has led to a problem which facilitates a fast numerical solution. This fast solution procedure has been obtained using a multigrid method.

Table I. Details of method for problem (i)

Work Units	$\bar{\lambda}$	$\ error\ _2$
13	0.49	0.45-4

Table II. Details of method for problem (ii)

m	n	Work Units	$\bar{\lambda}$	$\ error\ _2$
1	1	14	0.52	0.43-3
2	2	14	0.51	0.17-2
3	3	14	0.51	0.37-2
4	4	14	0.51	0.66-2
5	5	12	0.47	0.10-1
7	7	11	0.43	0.20-1
3	7	12	0.46	0.14-1

Table III. Details of method for problem (iii)

m	$\alpha$	Work Units	$\bar{\lambda}$	$\ error\ _2$
1	0.4	11	0.45	0.12-2
1	4.0	11	0.45	0.74-3
1	40.0	12	0.46	0.92-2

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